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The value function in a problem of chemotherapy of a malignant tumor growing according to the Gompertz law. ⋆

Nina N. Subbotina * **Natalia G. Novoselova** **

* *N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences (IMM UB RAS) 16 S.Kovalevskaya str., Yekaterinburg, 620990 Russia; Ural Federal University named after the first President of Russia B.N.Yeltsin, 620002, 19 Mira str., Yekaterinburg, Russia (e-mail: subb@uran.ru)*

** *N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences (IMM UB RAS) 16 S.Kovalevskaya str., Yekaterinburg, 620990 Russia; Ural Federal University named after the first President of Russia B.N.Yeltsin, 620002, 19 Mira str., Yekaterinburg, Russia (e-mail: n.g.novoselova@gmail.com)*

Abstract: In this paper a construction of the value function is obtained in the problem of chemotherapy of a malignant tumor growing according to the Gompertz law, when a therapy function has two maxima. The aim of therapy is to minimize the number of tumor cells at the given final instance.

Keywords: Optimal control, Rankine-Hugoniot line, Hamilton-Jacobi-Bellman equation, Cauchy method of characteristics.

1. INTRODUCTION

In this paper, a model of chemotherapy of a tumor growing according to the non-linear Gompertz law is considered. It is assumed that the function of therapy, describing the effect of the drug on tumor cells, has two maxima, in contrast to the case considered in the work (Bratus and Chumerina (2008)), when the nonmonotonic function of therapy has one maximum. The aim of this therapy is to minimize the number of tumor cells at the given final instance.

In the papers (Bratus and Chumerina (2008); Novoselova and Subbotina (2017)) the therapy models were considered when the term $g(m)$ defining the law of tumor growth is equal to zero for the chemotherapy model, and the therapy function is nonmonotonic. In work (Bratus and Chumerina (2008)) the value function of the considered problem is constructed for the case of the therapy function with one maximum. The construction of the value function in the case of the therapy function with two maxima is obtained in the work (Novoselova and Subbotina (2017)). This paper essentially relies on the results of the work (Novoselova and Subbotina (2017)).

In this paper, the value function is constructed in the problem of tumor chemotherapy, where a tumor grows according to the nonlinear Gompertz law with the therapy function having two maxima. This task is reduced to piecewise smooth function obtained with the help of the Cauchy characteristics method for the auxiliary linear Hamilton-Jacobi-Bellman equations. The justification of this constriction is based on results of the theory of generalized solutions of the Hamilton-Jacobi equation (Subbotin (1995); Subbotina et al. (2013)).

2. DYNAMICS

Let m be quantity of malignant cells, h be quantity of drug, $f(h)$ be a therapy function that describes the effect of drug on tumor cells, $u(t)$ be a restricted control.

The process of interaction between tumor cells and the drug is described by the following known model suggested by Bratus and Chumerina (2008), where $t \in [0, T]$:

$$\begin{aligned} \frac{dm}{dt} &= g(m) - \gamma m f(h),\ m(t_0) = m_0,\ \gamma - const > 0 \\ \frac{dh}{dt} &= -\alpha h + u(t),\ h(t_0) = h_0,\ \alpha - const > 0. \end{aligned} \tag{1}$$

Here $g(m) = r\,m - \theta\,m\,ln(m)$ is the Gompertz law, $r, \theta - const > 0$. Let M be the maximum quantity of malignant cells in the body compatible with life, L be the maximum quantity of drug in the body, Q be the maximum quantity

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of drug injected into the tumor per unit time, T be the given final instance,

$$t_0 \in [0,T], \quad 0 < m_0 < M, \quad 0 \le h_0 \le L.$$

We assumed that the quantity of drug injected into the tumor per unit time is restricted:

$$0 \le u(t) \le Q \tag{2}$$

3. THE THERAPY FUNCTION

We consider a piecewise monotone, continuously differentiable therapy function $f(h)$ with the following properties:
$A1$. $\{f(h) > 0,\ 0 < h < L,\} \bigcup \{f(h) \equiv 0,\ h \le 0, h \ge L\}$ and its derivative $f'(h) = \dfrac{df(h)}{dh}$ has 3 different real roots:

$$0 < \hat{h}_1 < \hat{h}_2 < \hat{h}_3 \le L,\ f'(\hat{h}_i) = 0.$$

$A2$. If $h < \hat{h}_1$ then $f'(h) > 0$ and if $h > \hat{h}_3$ then $f'(h) < 0$.
$A3$. $0 < \alpha\hat{h}_i < Q, \quad i = 1,2,3.$
$A4$. $f(\hat{h}_1) = f(\hat{h}_3)$.

Suppose that conditions $A1 - A4$ are satisfied in problem (1),(2). We investigate the situation when:

$$\{f'(h) < 0, h \in (\hat{h}_1, \hat{h}_2)\} \bigcup \{f'(h) > 0, h \in (\hat{h}_2, \hat{h}_3)\}. \tag{3}$$

Conditions (3) and $A2$ imply that roots $\hat{h}_1$ and $\hat{h}_3$ are points of maxima and root $\hat{h}_2$ is the point of mimima of therapy function $f(h)$.

We consider admissible controls as piecewise constant functions:

$$u(\cdot) : [t_0, T] \mapsto [0, Q].$$

4. STATEMENT

The aim of the optimal control problem is minimization of the terminal cost function:

$$\sigma(m(T)) = m^2(T; t_0, m_0, h_0, u(\cdot)) \to \inf_{u(\cdot)}, \tag{4}$$

where $m(t) = m(t; t_0, m_0, h_0, u(\cdot)),\ t \in [t_0, T]$ is the solution of the system (1) with initial conditions (t_0, m_0, h_0), generated under the influence of an admissible control $u(t)$. It is easy to verify that $m(t)$ has the following form

$$m(t) = m_0^{-\theta(t-t_0)} exp(\int_{t_0}^{t} e^{-\theta(t-\tau)}(r - \gamma f(h(\tau)))d\tau), \tag{5}$$

where $h(t) = h(t; t_0, h_0, u(\cdot))$ is the solution of the second equation of the system (1).

5. VALUE FUNCTION VAL

We introduce the value function in problem (1), (2), (4).This function associates the optimal result $Val(t_0, h_0, m_0)$ each initial state of the system $(t_0, h_0, m_0) \in [t_0, T] \times [0, L] \times [0, M]$.

From the equations of dynamics (1) we get that

$$\sigma(m(T)) = m^2(T; t_0, m_0, h_0, u(\cdot)) =$$
$$m_0^{-2\theta(t-t_0)} exp(2\int_{t_0}^{t} e^{-\theta(t-\tau)}(r - \gamma f(h(\tau)))d\tau),$$

where $h(t) = h(t; t_0, h_0, u(\cdot))$ is the solution of the second equation of the system (1).

It is not difficult to see that the equality holds:

$$Val(t_0, h_0, m_0) = m_0^{-2\theta(t-t_0)} e^{2r\int_{t_0}^{t} e^{-\theta(t-\tau)}d\tau} e^{-2\gamma V(t_0, h_0)}, \tag{6}$$

where $V(t_0, h_0)$ is the optimal result in the following reduced optimal control problem:

$$\frac{dh}{dt} = -\alpha h + u(t), \quad h(t_0) = h_0 \tag{7}$$

$$J_{t_0,h_0}(u(\cdot)) = \int_{t_0}^{T} e^{-\theta(T-\tau)} f(h(t; t_0, h_0, u(\cdot)))dt \to \sup_{u(\cdot)}, \tag{8}$$

The value function in the problem (7), (8) has the form $(t_0, h_0) \mapsto V(t_0, h_0) = \sup\limits_{u(\cdot)} J_{t_0,h_0}(u(\cdot)), \forall (t_0, h_0) \in [0, T] \times R.$

As is known (Subbotina et al. (2013); Subbotin (1995)) the value function is the minimax or viscosity generalized solution of the Hamilton-Jacobi-Bellman equation for the following Cauchy problem:

$$\begin{aligned} &\frac{\partial V}{\partial t} - \alpha h \frac{\partial V}{\partial h} + e^{-\theta(T-t)} f(h) + \max_{u \in [0,Q]} u \frac{\partial V}{\partial h} = 0, \\ &V(T, h) = 0. \end{aligned} \tag{9}$$

At points where $V(t, h)$ is differentiable, the value function satisfies this equation. At points where $V(t, h)$ is not differentiable, its superdifferential is nonempty, $D^- V(t, h) \neq \emptyset$ (Subbotin (1995)) and the following condition is satisfied:

$$\begin{aligned} &\forall (s_t, s_h) \in D^- V(t, h) \Rightarrow \\ &s_t - \alpha h s_h + f(h) + \max_{u \in [0,Q]} u s_h \le 0. \end{aligned} \tag{10}$$

6. CONSTRUCTION OF THE VALUE FUNCTION V

Let us construct a continuous function $\varphi(t, h)$ on the set $\Pi_T = \{[0, T] \times R\}, (t, h) \in \Pi_T$ and show that it is the value function $V(t, h)$ in the reduced problem (7), (8).

6.1 Functions $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$

Let us define the function $\varphi_1(\cdot)$ on the set

$$G_1 = \{(t, h) : t \in [0, T), h = \hat{h}_1\}.$$

As noted above $\hat{h}_1$ is the maximum of the therapy function. For any initial state $(t_0, \hat{h}_1) \in G_1$ the optimal behavior is $h(t) = h(t; t_1, \hat{h}_1, u^0(\cdot)) \equiv \hat{h}_1$, which can be achieved under the control $u^0(t) \equiv \alpha\hat{h}_1$ satisfying condition $A3$. The optimal result of problem (7),(8) at the point $(t_1, \hat{h}_1) \in G_1$ is equal to the value:

$$
\begin{aligned}
&V(t_1, h_0) = J_{t_1,h_0}(u^0(\cdot)) = \\
&= \int_{t_1}^{T} e^{-\theta(T-t)} f(h(t; t_1, h_0, u^0(\cdot))dt = \\
&= \frac{1}{\theta} f(\hat{h}_1)(1 - e^{-\theta(T-t_1)}).
\end{aligned}
\tag{11}
$$

We solve the following characteristic system with boundary conditions:

$$
\begin{aligned}
&\frac{dh}{dt} = -\alpha h + \alpha \hat{h}_1,\ h(T) = \hat{h}_1, \\
&\frac{ds_h}{dt} = \alpha s_h - e^{-\theta(T-t)} f'(\hat{h}_1),\ s_h(T) = 0, \\
&\frac{dz}{dt} = -e^{-\theta(T-t)} f(\hat{h}_1),\ z(T) = 0.
\end{aligned}
$$

For any points $(t_1, \hat{h}_1) \in G_1$ we obtain the solution:

$$
\begin{aligned}
&h(t_1) = \hat{h}_1, \\
&s_h(t_1) = 0, \\
&z(t_1) = \frac{1}{\theta} f(\hat{h}_1)(1 - e^{-\theta(T-t_1)}),
\end{aligned}
$$

and define $\varphi_1(t_1, \hat{h}_1)$ and $\frac{\partial \varphi_1}{\partial h}(t_1, \hat{h}_1)$ for any $t_1 \in [0, T]$ as:

$$
\begin{aligned}
&\varphi_1(t_1, \hat{h}_1) = z(t_1) \equiv \frac{1}{\theta} f(\hat{h}_1)(1 - e^{-\theta(T-t_1)}), \\
&\frac{\partial \varphi_1}{\partial h}(t_1, \hat{h}_1) = s_h(t_1) = 0.
\end{aligned}
\tag{12}
$$

According to (11) and (12) we get on the set G_1:

$$V(t_1, \hat{h}_1) = \varphi_1(t_1, \hat{h}_1).$$

Similar arguments hold for the function $\varphi_2(\cdot)$ on the set $G_2 = \{(t, h) : t \in [0, T), h = \hat{h}_3\}$, where $\varphi_2(t_2, \hat{h}_3)$ and $\frac{\partial \varphi_2}{\partial h}(t_2, \hat{h}_3)$ for all $t_2 \in [0, T]$ are defined as:

$$
\begin{aligned}
&\varphi_2(t_2, \hat{h}_1) = z(t_2) \equiv \frac{1}{\theta} f(\hat{h}_3)(1 - e^{-\theta(T-t_2)}), \\
&\frac{\partial \varphi_2}{\partial h}(t_2, \hat{h}_3) = s_h(t_2) = 0.
\end{aligned}
$$

Then we get that $V(t_2, \hat{h}_3) = \varphi_2(t_2, \hat{h}_3)$ on the set G_2.

6.2 Functions $\varphi_3(\cdot)$ and $\varphi_4(\cdot)$

Let us define the function $\varphi_3(\cdot)$ in the area

$$\Pi_1 = [0, T) \times [0, \hat{h}_1].$$

The set G_1 introduced earlier is a part of the border of this area Π_1. We assume that the relations are true on the set G_1:

$$
\begin{aligned}
&\varphi_3(t_1, \hat{h}_1) = \varphi_1(t_1, \hat{h}_1) = V(t_1, \hat{h}_1), \\
&\frac{\partial \varphi_3}{\partial h}(t_1, \hat{h}_1) = 0.
\end{aligned}
\tag{13}
$$

Consider the set $G_3 = \{(t, h) : t = T, h \in [0, \hat{h}_1]\}$, which is another part of the boundary of the area Π_1, where we assume

$$
\begin{aligned}
&\varphi_3(T, h) = V(T, h) = J_{t,h}(u(\cdot)) = \\
&\int_{T}^{T} e^{-\theta(T-t)} f(h(t; T, h, u(\cdot))dt = 0, \quad \frac{\partial \varphi_3}{\partial h}(T, h) = 0.
\end{aligned}
\tag{14}
$$

We construct the classical solution of the linear Hamilton-Jacobi equation in the area Π_1

$$
\begin{aligned}
&\frac{\partial \varphi_3}{\partial t} + H^Q\Big(h, \frac{\partial \varphi_3}{\partial h}\Big) := \\
&\frac{\partial \varphi_3}{\partial t} - \alpha h \frac{\partial \varphi_3}{\partial h} + e^{-\theta(T-t)} f(h) + Q \frac{\partial \varphi_3}{\partial h} = 0
\end{aligned}
\tag{15}
$$

with the boundary condition defined by (13) on the set G_1 and defined by (14) on the set G_3. We construct the solution of the following characteristic system:

$$
\begin{aligned}
&\frac{dh}{dt} = -\alpha h + Q, \\
&\frac{ds_h}{dt} = \alpha s_h - e^{-\theta(T-t)} f'(h), \\
&\frac{dz}{dt} = -e^{-\theta(T-t)} f(h),
\end{aligned}
$$

with boundary conditions, when $(t_1, h(t_1) = \hat{h}_1) \in G_1$:

$$
\begin{aligned}
&h(t_1) = \hat{h}_1, \\
&s_h(t_1) = 0, \\
&z(t_1) = \frac{1}{\theta} f(\hat{h}_1)(1 - e^{-\theta(T-t_1)}),
\end{aligned}
$$

and with boundary conditions, when $(T, h(T)) \in G_3$:

$$
\begin{aligned}
&h(T) = \gamma_1, \gamma_1 \in (0, \hat{h}_1) \\
&s_h(T) = 0, \\
&z(T) = 0.
\end{aligned}
$$

At points $(t_0, h_0) \in \Pi_1$ on the state characteristics with boundary conditions for G_1 the solution has the form:

$$
\begin{aligned}
&h_0 = h(t_0) = (\hat{h}_1 - \frac{Q}{\alpha}) e^{\alpha(t_1 - t_0)} + \frac{Q}{\alpha}, \\
&s_h(t_0) = -e^{\alpha t_0 - \theta T} \int_{t_0}^{t_1} e^{\tau(\theta - \alpha)} f'(h(\tau)) d\tau, \\
&z(t_0) = \frac{1}{\theta} f(\hat{h}_1)(1 - e^{-\theta(T-t_1)}) + \int_{t_0}^{t_1} e^{-\theta(T-\tau)} f(h(\tau)) d\tau.
\end{aligned}
$$

At points $(t_0, h_0) \in \Pi_1$ on the state characteristics with boundary conditions for G_3 the solution has the form:

$$
\begin{aligned}
&h_0 = h(t_0) = (\gamma_1 - \frac{Q}{\alpha}) e^{\alpha(T - t_0)} + \frac{Q}{\alpha}, \\
&s_h(t_0) = -e^{\alpha t_0 - \theta T} \int_{t_0}^{T} e^{\tau(\theta - \alpha)} f'(h(\tau)) d\tau, \\
&z(t_0) = \int_{t_0}^{T} e^{-\theta(T-\tau)} f(h(\tau)) d\tau.
\end{aligned}
\tag{16}
$$

According to the Cauchy method, the solution of boundary value problem for equation (15) in the area Π_1 has the form

$$h_0 = h(t_0), \quad \varphi_3(t_0, h_0) = z(t_0).$$

Remark 1. It follows from condition $A2$, namely $f'(h) > 0$, $h \in [0, \hat{h}_1]$, that $s_h(t_0) = \dfrac{\partial \varphi_3(t_0, h_0)}{\partial h} > 0$ at the interior points of the area Π_1. Consequently, the function $\varphi_3(t, h)$ in the area Π_1 satisfies the equation:

$$
\begin{aligned}
&\frac{\partial \varphi_3(t, h)}{\partial t} - \alpha h \frac{\partial \varphi_3(t, h)}{\partial h} + e^{-\theta(T-t)} f(h) + \\
&Q \frac{\partial \varphi_3(t, h)}{\partial h} = \frac{\partial \varphi_3(t, h)}{\partial t} - \alpha h \frac{\partial \varphi_3(t, h)}{\partial h} + \\
&e^{-\theta(T-t)} f(h) + \max_{u \in [0, Q]} u \frac{\partial \varphi_3(t, h)}{\partial h} = 0,
\end{aligned}
$$

and $s_h(t_0) = \dfrac{\partial \varphi_3(t_0,h_0)}{\partial h} = 0$ at $(t_0,h_0) \in G_1 \bigcup G_3$.

Similar arguments are valid in the area $\Pi_2 = [0,T) \times [\hat{h}_3, L]$ for the function $\varphi_4(\cdot)$ which is the classical solution of the following boundary value problem:

$$\frac{\partial \varphi_4(t,h)}{\partial t} - \alpha h \frac{\partial \varphi_4(t,h)}{\partial h} + e^{-\theta(T-t)} f(h) + 0 \cdot \frac{\partial \varphi_4(t,h)}{\partial h} = \frac{\partial \varphi_4(t,h)}{\partial t} - \alpha h \frac{\partial \varphi_4(t,h)}{\partial h} + e^{-\theta(T-t)} f(h) + \max_{u\in[0,Q]} u \frac{\partial \varphi_4(t,h)}{\partial h} = 0,$$

with boundary conditions for $(t_2, \hat{h}_3) \in G_2$:

$$\varphi_4(t_2,\hat{h}_3) = z(t_2), \quad \frac{\partial \varphi_4(t_2,\hat{h}_3)}{\partial h} = 0$$

and with boundary conditions, when $(T, \gamma_2) \in G_4$:

$$\varphi_4(T,\gamma_2) = z(T), \quad \frac{\partial \varphi_4(T,\gamma_2)}{\partial h} = 0$$

According to the Caushy method of characteristics, this solution is constructed with the help of the corresponding characteristic system in the area $\Pi_2 \ni (t_0, h_0)$ and has the form

$$h_0 = h(t_0), \quad \varphi_4(t_0,h_0) = z(t_0).$$

Remark 2. It follows from condition $A2$, namely $f'(h) < 0$, $h \in [\hat{h}_3, L]$, that $s_h(t_0) = \dfrac{\partial \varphi_4(t_0,h_0)}{\partial h} < 0$ at the interior points of the area Π_2. Consequently, the function $\varphi_4(t,h)$ in the area Π_2 satisfies the equation:

$$\frac{\partial \varphi_4(t,h)}{\partial t} - \alpha h \frac{\partial \varphi_4(t,h)}{\partial h} + e^{-\theta(T-t)} f(h) = \frac{\partial \varphi_4(t,h)}{\partial t} - \alpha h \frac{\partial \varphi_4(t,h)}{\partial h} + e^{-\theta(T-t)} f(h) + \max_{u\in[0,Q]} u \frac{\partial \varphi_4(t,h)}{\partial h} = 0,$$

and $\dfrac{\partial \varphi_4(t_0,h_0)}{\partial h} = 0$ at $(t_0,h_0) \in G_2 \bigcup G_4$.

6.3 Functions $\varphi_5(\cdot)$ and $\varphi_6(\cdot)$

Let us define the function $\varphi_5(\cdot)$ in the area

$$\Pi = [0,T] \times [\hat{h}_1, \hat{h}_3].$$

The set G_1 introduced earlier is a part of the border of this area Π. We assume that the relations are true on the set G_1:

$$\varphi_5(t_0,\hat{h}_1) = \varphi_1(t_0,\hat{h}_1) = V(t_0,\hat{h}_1), \quad \frac{\partial \varphi_5}{\partial h}(t_0,\hat{h}_1) = 0. \tag{17}$$

Consider the set $G_5 = \{(t,h) : t = T, h \in [\hat{h}_1, \hat{h}_2]\}$, which is a part of the border area Π, where we assume

$$\varphi_5(T,h) = V(T,h) = J_{t,h}(u(\cdot)) = \int_T^T e^{-\theta(T-t)} f(h(t;T,h,u(\cdot))dt = 0, \ \frac{\partial \varphi_5}{\partial h}(T,h) = 0. \tag{18}$$

We construct the classical solution of the linear Hamilton-Jacobi equation in the area Π

$$\frac{\partial \varphi_5}{\partial t} + H^0\left(h, \frac{\partial \varphi_5}{\partial h}\right) := \frac{\partial \varphi_5}{\partial t} - \alpha h \frac{\partial \varphi_5}{\partial h} + e^{-\theta(T-t)} f(h) + 0 \cdot \frac{\partial \varphi_5}{\partial h} = 0 \tag{19}$$

with the boundary condition defined by (17) on the set G_1 and defined by (18) on the set G_5. We construct the solution of the following characteristic system:

$$\frac{dh}{dt} = -\alpha h, \quad \frac{ds_h}{dt} = \alpha s_h - e^{-\theta(T-t)} f'(h), \quad \frac{dz}{dt} = -e^{-\theta(T-t)} f(h)$$

with boundary conditions, when $(t_1, h(t_1) = \hat{h}_1) \in G_1$:

$$h(t_1) = \hat{h}_1, \quad s_h(t_1) = 0, \quad z(t_1) = \frac{1}{\theta} f(\hat{h}_1)(1 - e^{-\theta(T-t_1)}),$$

and with boundary conditions, when $(T, h(T)) \in G_5$:

$$h(T) = \xi_1, \xi_1 \in (\hat{h}_1, \hat{h}_2), \quad s_h(T) = 0, \quad z(T) = 0.$$

For points $(t_0,h_0) \in \Pi$ lying on the graphs of the state characteristics $(t, \tilde{h}(t))$ with boundary conditions on G_1 the solution of this system has the form:

$$h_0 = \tilde{h}(t_0) = \hat{h}_1 e^{\alpha(t_1 - t_0)},$$

$$s_h^5(t_0) = e^{\alpha t_0 - \theta T} \int_{t_0}^{t_1} e^{\tau(\theta-\alpha)} f'(\tilde{h}(\tau)) d\tau,$$

$$z^5(t_0) = \frac{1}{\theta} f(\hat{h}_1)(1 - e^{-\theta(T-t_1)}) + \int_{t_0}^{t_1} e^{-\theta(T-\tau)} f(\tilde{h}(\tau)) d\tau.$$

For points $(t_0,h_0) \in \Pi$ lying on the state of the phase characteristics $(t, \tilde{h}(t))$ with boundary conditions on G_5 the solution has the form:

$$h_0 = \tilde{h}(t_0) = \xi_1 e^{\alpha(T - t_0)},$$

$$s_h^5(t_0) = e^{\alpha t_0 - \theta T} \int_{t_0}^{T} e^{\tau(\theta-\alpha)} f'(\tilde{h}(\tau)) d\tau,$$

$$z^5(t_0) = \int_{t_0}^{T} e^{-\theta(T-\tau)} f(\tilde{h}(\tau)) d\tau,$$

According to the Cauchy method of characteristics, we construct the solution of equation (19) with boundary conditions on $G_1 \bigcup G_5$. In the subarea of the area Π covered by graphs of state characteristics $(t, \tilde{h}(t))$ the solution has the form

$$\tilde{h}(t_0) = h_0, \quad z^5(t_0) = \varphi_5(t_0,h_0), \quad s_h^5(t_0) = \frac{\partial \varphi_5(t_0,h_0)}{\partial h}.$$

Remark 3. Using (3) we obtain that the condition $\dfrac{\partial \varphi_5(t,h)}{\partial h} = s_h^5(t) < 0$ is true at the points of the area $(t,h) \in [0,T) \times (\hat{h}_1, \hat{h}_2]$ covered by graphs of state characteristics $(t, \tilde{h}(t))$ with boundary conditions on $G_1 \bigcup G_5$.

Similar arguments are valid in the area Π for the function $\varphi_6(\cdot)$ which is the classical solution of the following boundary value problem:

$$\frac{\partial\varphi_6(t,h)}{\partial t}-\alpha h\frac{\partial\varphi_6(t,h)}{\partial h}+e^{-\theta(T-t)}f(h)+$$
$$Q\cdot\frac{\partial\varphi_6(t,h)}{\partial h}=\frac{\partial\varphi_6(t,h)}{\partial t}-\alpha h\frac{\partial\varphi_6(t,h)}{\partial h}+$$
$$e^{-\theta(T-t)}f(h)+\max_{u\in[0,Q]}u\frac{\partial\varphi_6(t,h)}{\partial h}=0,$$

with boundary conditions for $(t_2,\hat{h}_3)\in G_2$:

$$\varphi_6(t_2,\hat{h}_3)=z(t_2),\quad \frac{\partial\varphi_6(t_2,\hat{h}_3)}{\partial h}=0$$

and with boundary conditions, when $(T,\xi_2)\in G_6$:

$$\varphi_6(T,\xi_2)=z(T),\quad \frac{\partial\varphi_6(T,\xi_2)}{\partial h}=0$$

According to the Caushy method of characteristics, this solution is constructed with the help of the corresponding characteristic system in the area $\Pi\ni(t_0,h_0)$ and has the form

$$\bar{h}(t_0)=h_0,\quad z^6(t_0)=\varphi_6(t_0,h_0),\quad s_h^6(t_0)=\frac{\partial\varphi_6(t_0,h_0)}{\partial h}.$$

Remark 4. Using (3) we obtain that the condition $\frac{\partial\varphi_6(t,h)}{\partial h}=s_h^6(t)>0$ is true at the points of the area $(t,h)\in[0,T)\times(\hat{h}_1,\hat{h}_2]$ covered by graphs of state characteristics $(t,\bar{h}(t))$ with boundary conditions on $G_2\bigcup G_6$.

7. CONSTRUCTION OF THE FUNCTION $\varphi(\cdot)$ IN THE AREA Π

The constructed functions φ_5 and φ_6 intersect in the strip Π. Points with the same values of the functions form a line $\Gamma=\{(t,x(t)):t\in[0,T],x(T)=\hat{h}_2\}$. Continuity of gluing together these functions and arguments similar to the arguments in the paper Goritskiy et al. (1999); Subbotina et al. (2013) imply that the equation of the gluing line Γ satisfies the Rankin-Hugoniot condition of the form:

$$\frac{dx}{dt}=\frac{H^Q(x,s_h^6)-H^0(x,s_h^5)}{s_h^6-s_h^5}=-\alpha x(t)+Q\frac{s_h^6}{s_h^6-s_h^5}\quad(20)$$

with the boundary condition $x(T)=\hat{h}_2$.

In the area Π we introduce the function $\varphi(t,h)$ of the form:

$$\varphi(t,h)=\begin{matrix}\varphi_5(t,h), & (t,h)\in\Pi_3=[0,T)\times(\hat{h}_1,x(t)],\\ \varphi_6(t,h), & (t,h)\in\Pi_4=[0,T)\times[x(t),\hat{h}_3),\\ 0, & t=T,h\in[\hat{h}_1,\hat{h}_3]\end{matrix}$$

Using the schemes of proof, given in the work Novoselova and Subbotina (2017), for the case when $g(m)=0$ we obtain the following assertions:

Assertion 1. For all points $(t,h)\in\Pi$, where φ_5 and φ_6 are simultaneously defined, it is true:

$$\varphi(t,h)=\max\{\varphi_5(t,h),\varphi_6(t,h)\}.\quad(21)$$

Corollary 1. According to the assertion 1 we get:

$$\frac{\partial\varphi(t,h)}{\partial h}=\frac{\partial\varphi_5(t,h)}{\partial h}=s_h^5(t,h)<0,\quad(t,h)\in\Pi_3,$$
$$\frac{\partial\varphi(t,h)}{\partial h}=\frac{\partial\varphi_6(t,h)}{\partial h}=s_h^6(t,h)>0,\quad(t,h)\in\Pi_4.$$

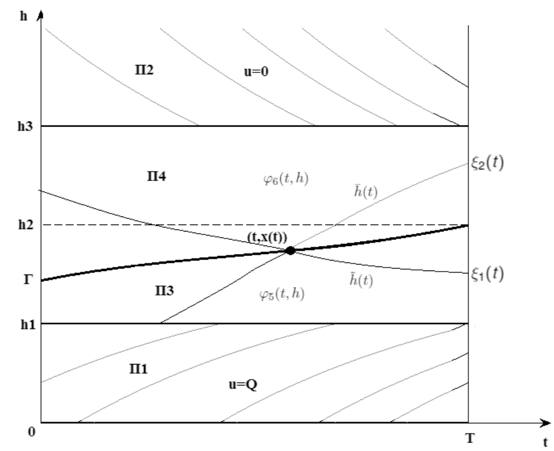

Fig. 1. Construction of function $\varphi(\cdot)$

Consequently, in the area $\Pi\backslash\Gamma$ the function $\varphi(t,h)$ satisfies the equation:

$$\frac{\partial\varphi(t,h)}{\partial t}-\alpha h\frac{\partial\varphi(t,h)}{\partial h}+$$
$$e^{-\theta(T-t)}f(h)+max_{u\in[0,Q]}(u\frac{\partial\varphi(t,h)}{\partial h})=0,$$

and it is superdifferentiable at the points of the curve Γ. Its superdifferential has the form (Subbotin (1995)):

$$D^-\varphi(t,h)=co\Big\{\Big(\frac{\partial\varphi_5}{\partial t},\frac{\partial\varphi_5}{\partial h}\Big);\Big(\frac{\partial\varphi_6}{\partial t},\frac{\partial\varphi_6}{\partial h}\Big)\Big\},\quad(22)$$

where the symbol *co* denotes a convex hull.

Assertion 2. The following inequality holds for any point $(\tilde{t},x(\tilde{t}))$ on the curve Γ and for all elements (s_t,s_h) from the superdifferential $D^-\varphi t,h)$:

$$s_t-\alpha hs_h+e^{-\theta(T-t)}f(h)+\max_{u\in[0,Q]}us_h\leq 0.\quad(23)$$

8. MAIN RESULT

Theorem 1. The function $\varphi(\cdot)$ of the form:

$$\varphi(t,h)=\begin{matrix}\varphi_1, & (t,h)\in G_1,\\ \varphi_2, & (t,h)\in G_2,\\ \varphi_3, & (t,h)\in\Pi_1,\\ \varphi_4, & (t,h)\in\Pi_2,\\ \varphi_5, & (t,h)\in\Pi_3,\\ \varphi_6, & (t,h)\in\Pi_4,\end{matrix}\quad(24)$$

coincides with the value function $V(t,h)$ in problem (7), (8) on Π_T.

The picture (1) illustrates the construction of function $\varphi(\cdot)$.

Proof 1. According the above constructions the following conditions

$$\varphi_3(t,\hat{h}_1)=\varphi_5(t,\hat{h}_1)=\varphi(t,\hat{h}_1),$$
$$\frac{\partial\varphi_3}{\partial h}(t,\hat{h}_1)=\frac{\partial\varphi_5}{\partial h}(t,\hat{h}_1)=0.$$

hold at the point $\hat{h}_1$, for any $t\in[0,T]$. Equations (15), (19) imply that the equalities are true at the point $(t,\hat{h}_1)$:

$$\frac{\partial\varphi_3}{\partial t}(t,\hat{h}_1)=\frac{\partial\varphi_5}{\partial t}(t,\hat{h}_1)=0.$$

So the functions $\varphi_3(\cdot), \varphi_5(\cdot)$ glue together smoothly and the following equation for $\varphi(t, \hat{h}_1)$ is valid:

$$\frac{\partial \varphi(t,\hat{h}_1)}{\partial t} - \alpha h \frac{\partial \varphi(t,\hat{h}_1)}{\partial h} + e^{-\theta(T-t)} f(h) + max_{u\in[0,Q]} \left(u \frac{\partial \varphi(t,\hat{h}_1)}{\partial h} \right) = 0.$$

Similarly, the functions $\varphi_4(\cdot), \varphi_6(\cdot)$ glue together smoothly and the following equation for $\varphi(t, \hat{h}_3)$ is true:

$$\frac{\partial \varphi(t,\hat{h}_3)}{\partial t} - \alpha h \frac{\partial \varphi(t,\hat{h}_3)}{\partial h} + e^{-\theta(T-t)} f(h) + max_{u\in[0,Q]} \left(u \frac{\partial \varphi(t,\hat{h}_3)}{\partial h} \right) = 0.$$

The functions $\varphi_5(t, h)$ and $\varphi_6(t, h)$ are equal on Γ, but the gradients of these functions do not coincide. It means that these functions are not glued together smoothly on the curve Γ.

It follows from the remarks 3, 4 and the corollary 1 that the function $\varphi(t, h)$ of the form (24) is continuously differentiable in the area $(t, h) \in \Pi_T \setminus \Gamma$ and satisfies the equation:

$$\frac{\partial \varphi(t,h)}{\partial t} - \alpha h \frac{\partial \varphi(t,h)}{\partial h} + e^{-\theta(T-t)} f(h) + max_{u\in[0,Q]} \left(u \frac{\partial \varphi(t,h)}{\partial h} \right) = 0. \tag{25}$$

This function is superdifferentiable at points $(\tilde{t}, x(\tilde{t})) \in \Gamma$. All elements of the superdifferential satisfy the inequality:

$$s_t(\tilde{t}) - \alpha x(\tilde{t}) s_h(\tilde{t}) + e^{-\theta(T-\tilde{t})} f(x(\tilde{t})) + \max_{u\in[0,Q]} u s_h(\tilde{t}) \leq 0.$$

Then, according to the theory of minimax solutions (Subbotin (1995)), the constructed function $\varphi(t, h)$ is the minimax solution of the equation (25) with the boundary condition $\varphi(T, h) = 0$ and coincides with the value function $V(t, h)$ in the problem (7), (8). Q.E.D.

Thus, using the construction $V(t, h)$, the value function $Val(t_0, m_0, h_0)$ in the initial problem (4) is defined according to (6).

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